

Dynamical symmetries of two-dimensional systems in relativistic quantum mechanics

Fu-Lin Zhang,* Ci Song, and Jing-Ling Chen†

Theoretical Physics Division, Chern Institute of Mathematics,
Nankai University, Tianjin 300071, People's Republic of China

(Dated: October 13, 2008)

The two-dimensional Dirac Hamiltonian with equal scalar and vector potentials has been proved commuting with the deformed orbital angular momentum L . When the potential takes the Coulomb form, the system has an $SO(3)$ symmetry, and similarly the harmonic oscillator potential possesses an $SU(2)$ symmetry. The generators of the symmetric groups are derived for these two systems separately. The corresponding energy spectra are yielded naturally from the Casimir operators. Their non-relativistic limits are also discussed.

PACS numbers: 03.65.-w; 02.20.-a; 21.10.sf; 31.30.jx

Dynamical symmetries have been known for a long time in both classical mechanics and quantum mechanics [1]. In classical mechanics, both the orbits of the Kepler problem and the isotropic oscillator are closed [2]. This feature suggests there are more constants of motion other than the orbital angular momentum. They have been shown as the Rung-Lenz vector [3, 4] in the Kepler problem and the second order tensors [5] in the isotropic oscillator. These conserved quantities generate the $SO(4)$ and $SU(3)$ Lie groups respectively. They are not geometrical but the symmetries in the phase space, and are called dynamical symmetries. These symmetries lead to an algebraic approach to determine the energy levels. Generally, the N -dimensional (ND) hydrogen atom has the $SO(N+1)$ and the isotropic oscillator has the $SU(N)$ symmetry.

In the relativistic quantum mechanics, the motion of spin- $\frac{1}{2}$ particle satisfies the Dirac equation. A dynamical symmetry does not exist in either the Dirac hydrogen atom or the Dirac oscillator. Our question is: Do there exist two Dirac systems which have the same dynamical symmetries as the non-relativistic hydrogen atom and isotropic oscillator separately? In both the non-relativistic limits of the two Dirac systems, there exists a spin-orbit coupling in the Hamiltonians [6, 7]. This suggests that the main reason of the breaking of dynamical symmetries is the spin-orbit coupling. Therefore, in the Dirac system which has the same dynamical symmetry as the non-relativistic hydrogen atom or the isotropic oscillator, the spin and orbital angular momentum should be conserved separately, and the potential takes the Coulomb or harmonic oscillator form.

Neither the spin nor the orbital angular momentum commutes with the Dirac Hamiltonian, even though in the free particle system. But, it has been shown that, in the Dirac system with equal scalar and vector potentials, the total angular momentum can be divided into

conserved orbital and spin parts as [8]

$$\vec{L} = \begin{bmatrix} \vec{l} & 0 \\ 0 & U_p \vec{l} U_P \end{bmatrix}, \quad \vec{S} = \begin{bmatrix} \vec{s} & 0 \\ 0 & U_p \vec{s} U_P \end{bmatrix}, \quad (1)$$

where $\vec{l} = \vec{r} \times \vec{p}$, $\vec{s} = \frac{\vec{\sigma}}{2}$ are the usual spin generators, $\vec{\sigma}$ are the Pauli matrices, and $U_p = U_p^\dagger = \frac{\vec{\sigma} \cdot \vec{p}}{p}$ is the helicity unitary operator [9]. The components of them form the $SU(2)$ Lie algebra separately. Then, the Dirac Hamiltonian, in the relativistic units, $\hbar = c = 1$, takes the form

$$H = \vec{\alpha} \cdot \vec{p} + \beta M + (1 + \beta) \frac{V(r)}{2}, \quad (2)$$

where $\vec{\alpha}$ and β are the Dirac matrices, and M is the mass. Ginocchio [10, 11] has proved that the Hamiltonian has the $U(3)$ symmetry in three-dimensional (3D) case, when the potential $V(r)$ takes the harmonic oscillator form. And the angular momentum given in (1) are three of the eight generators.

In this work we discuss the two-dimensional (2D) case. We choose $\alpha_1 = \sigma_1$, $\alpha_2 = \sigma_2$ and $\beta = \sigma_3$, and write the Hamiltonian in matrix form as

$$H = \begin{bmatrix} M + V(r) & p_1 - ip_2 \\ p_1 + ip_2 & -M \end{bmatrix}. \quad (3)$$

One can notice there is no the helicity unitary operator in the 2D system. So we introduce a 2D version definition of the conserved orbital angular momentum as

$$L = \begin{bmatrix} l & 0 \\ 0 & B^\dagger \frac{l}{p^2} B \end{bmatrix}, \quad (4)$$

where $B = p_1 - ip_2$, $B^\dagger = p_1 + ip_2$, and $l = x_1 p_2 - x_2 p_1$ is the usual orbital angular momentum. It is easy to prove the commutation relation $[L, H] = 0$.

The following question is: do there exist any other additional conserved quantities when the potentials $V(r)$ are of some spacial forms? We assume the constants of motion take the form as

$$Q = \begin{bmatrix} Q_{11} & Q_{12}B \\ B^\dagger Q_{21} & B^\dagger Q_{22}B \end{bmatrix}. \quad (5)$$

*Email:flzhang@mail.nankai.edu.cn

†Email:chenjl@nankai.edu.cn

The commutation relation $[Q, H] = 0$ requires the matrix elements must satisfy the equations:

$$\begin{aligned} Q_{12} &= Q_{21}, \\ [Q_{11}, V(r)] + [Q_{12}, p^2] &= 0, \\ [Q_{12}, V(r)] + [Q_{22}, p^2] &= 0, \\ Q_{11} &= Q_{12}(2M + V(r)) + Q_{22}p^2. \end{aligned} \quad (6)$$

They are the same as the 3D case [11] (this suggests $V(r)$ of the Eq. (1) in [11] should be $V(r)/2$). In the following paragraphs we will give the solutions of (6) with the Coulomb and harmonic oscillator potentials in turn.

Hydrogen atom. In non-relativistic hydrogen atom, the constants of motion are the orbital angular momentum l and two components of the Rung-Lenz vector

$$R_i = \frac{f_i}{2Mk} - \frac{x_i}{r}, \quad i = 1, 2, \quad (7)$$

where $f_1 = 2p_2l - ip_1$, $f_2 = -2p_1l - ip_2$, and k is the parameter in the Coulomb potential $V^h(r) = -\frac{k}{r}$. One can get the following relations easily

$$\begin{aligned} [f_i, p^2] &= 0, \quad \left[-\frac{x_i}{r}, V^h(r) \right] = 0, \\ \frac{1}{2Mk} [f_i, V^h(r)] + \frac{1}{2M} \left[-\frac{x_i}{r}, p^2 \right] &= 0. \end{aligned} \quad (8)$$

Therefore, we can obtain the solutions of (6) when the potential $V(r) = V^h(r) = -\frac{k}{r}$

$$Q_i^h = \begin{bmatrix} 2MR_i + \frac{kx_i}{r^2} & \left(-\frac{x_i}{r} \right) B \\ B^\dagger \left(-\frac{x_i}{r} \right) & B^\dagger \left(\frac{1}{k} \frac{f_i}{p^2} \right) B \end{bmatrix}, \quad i = 1, 2. \quad (9)$$

The commutation relations of the quantities are

$$\begin{aligned} [L, Q_1^h] &= iQ_2^h, \quad [L, Q_2^h] = -iQ_1^h, \\ [Q_1^h, Q_2^h] &= -i\frac{4}{k^2}(H_h^2 - M^2)L, \end{aligned} \quad (10)$$

and

$$(Q_1^h)^2 + (Q_2^h)^2 = \frac{H_h^2 - M^2}{k^2}(4L^2 + 1) + (H_h + M)^2, \quad (11)$$

where

$$H_h = \begin{bmatrix} M - \frac{k}{r} & p_1 - ip_2 \\ p_1 + ip_2 & -M \end{bmatrix}, \quad (12)$$

is the Dirac Hamiltonian in Eq. (3) with $V(r) = V^h(r)$. The deformed orbital angular momentum L and Q_i^h commute with H_h , $[L, H_h] = 0$, $[Q_i^h, H_h] = 0$.

These results show that the 2D Dirac system with equal scalar and vector potentials has the $SO(3)$ symmetry. The relations of the generators can also be used to solve the energy levels of this system. We define the normalized generators

$$\begin{aligned} A_1 &= \left[-\frac{4}{k^2}(H_h^2 - M^2) \right]^{-\frac{1}{2}} Q_1^h, \\ A_2 &= \left[-\frac{4}{k^2}(H_h^2 - M^2) \right]^{-\frac{1}{2}} Q_2^h, \\ A_3 &= L. \end{aligned} \quad (13)$$

Then,

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad (i, j, k = 1, 2, 3). \quad (14)$$

The $SO(3)$ Casimir operator is given by

$$C_{so3} = A_1^2 + A_2^2 + A_3^2 = j(j+1), \quad j = 0, 1, 2, \dots \quad (15)$$

Inserting (11) and (13) into (15), one can get the eigenvalues of the Hamiltonian H_h as

$$E_h^\pm = \frac{\pm n^2 - k^2}{n^2 + k^2} M, \quad n = 2j + 1 = 1, 3, 5, \dots \quad (16)$$

It takes the same form as the 3D case but different in the values of n [12].

When $M \rightarrow \infty$, $H \rightarrow M$, the non-relativistic limit of the energy levels is given by

$$E_h^+ \rightarrow M - \frac{2k^2}{n^2} M, \quad (17)$$

the second term of which agrees with the non-relativistic results [13]. We can also get the non-relativistic limits of the conserved quantities

$$H_h - M \rightarrow \frac{H_h^2 - M^2}{2M} \rightarrow \begin{bmatrix} \frac{p^2}{2M} - \frac{k}{r} & 0 \\ 0 & \frac{p^2}{2M} \end{bmatrix}, \quad (18)$$

$$\frac{Q_i^h}{2M} \rightarrow \begin{bmatrix} R_i & 0 \\ 0 & \frac{1}{2Mk} f_i \end{bmatrix}. \quad (19)$$

The upper-left elements of the above matrices are nothing but the non-relativistic hydrogen atom Hamiltonian and the Lung-Lenz vector, and the lower-right ones are their limits when $k \rightarrow 0$.

Harmonic oscillator. When the potential takes the harmonic oscillator form, $V(r) = V^o(r) = \frac{1}{2}M\omega^2r^2$, the Dirac Hamiltonian becomes

$$H_o = \begin{bmatrix} M + \frac{1}{2}M\omega^2r^2 & p_1 - ip_2 \\ p_1 + ip_2 & -M \end{bmatrix}. \quad (20)$$

To get the conserved quantities, we review some results in non-relativistic harmonic oscillator. The constants of motion in non-relativistic case are the orbital angular momentum $J_2 = \frac{l}{2}$ and the second order tensors

$$J_i = \frac{1}{2} \left(\frac{1}{M\omega} [pp]_i + M\omega [rr]_i \right), \quad i = 1, 3, \quad (21)$$

where $[rr]_1 = x_1x_2$, $[rr]_3 = \frac{x_1^2 - x_2^2}{2}$, $[pp]_1 = p_1p_2$ and $[pp]_3 = \frac{p_1^2 - p_2^2}{2}$. They satisfy

$$\begin{aligned} [[pp]_i, p^2] &= 0, \quad [[rr]_i, V^o(r)] = 0, \\ \frac{1}{2M\omega} [[pp]_i, V^o(r)] + \frac{\omega}{2} [[rr]_i, p^2] &= 0. \end{aligned} \quad (22)$$

Insert these relations into (6), we can get the conserved quantities in relativistic the harmonic oscillator potential as

$$Q_i^o = \begin{bmatrix} \frac{4}{\omega} J_i + [rr]_i V^o(r) & [rr]_i B \\ B^\dagger [rr]_i & \frac{2}{M\omega^2} B^\dagger \frac{[pp]_i}{p^2} B \end{bmatrix}, \quad i = 1, 3. \quad (23)$$

The commutation relations of L and Q_i^o are

$$\begin{aligned} [Q_1^o, L] &= i2Q_3^o, \quad [L, Q_3^o] = i2Q_1^o, \\ [Q_3^o, Q_1^o] &= i2\frac{2}{M\omega^2}(H_o + M)L, \end{aligned} \quad (24)$$

which show the constants of motion construct the $SU(2)$ Lie algebra.

The Casimir operator of the $SU(2)$ is

$$\begin{aligned} C_{su2} &= \frac{1}{4} \frac{M\omega^2}{2(H_o + M)} [(Q_1^o)^2 + (Q_3^o)^2] + \frac{L^2}{4} \quad (25) \\ &= s(s+1), \quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots \end{aligned}$$

Then, the eigenvalues of the Hamiltonian (20) are got as $E^- = -M$ or $E_n^+ = M$, which is the real root of the cubic equation

$$\frac{(E^+ + M)(E^+ - M)^2}{2Mw^2} - (n+1)^2 = 0, \quad (26)$$

where $n = 2s = 0, 1, 2, \dots$. It is coincided with the result in 3D case in [10].

When $M \rightarrow \infty$, $H \rightarrow M$, and the coefficient of elasticity $M\omega^2$ keeps unchangeably, Eq. (26) becomes the quadratic equation

$$(E^+ - M)^2 - (n+1)^2\omega^2 = 0, \quad (27)$$

which leads the non-relativistic energy levels of the 2D harmonic oscillator. At the same time, the operators give

$$H_o - M \rightarrow \begin{bmatrix} \frac{p^2}{2M} + \frac{M\omega^2}{2}r^2 & 0 \\ 0 & \frac{p^2}{2M} \end{bmatrix}, \quad (28)$$

$$\frac{\omega Q_i^o}{4} \rightarrow \begin{bmatrix} J_i & 0 \\ 0 & \frac{1}{2M\omega}[pp]_i \end{bmatrix}. \quad (29)$$

In conclusion, we have shown that the 2D Dirac systems with equal scalar and vector potentials have the $SO(3)$ symmetry with the Coulomb potential, and the $SU(2)$ symmetry for the harmonic oscillator potential. The nature of these symmetries are not geometrical but dynamical. Their Hamiltonian can be expressed in terms of the Casimir operators of the symmetry groups, which yield the energy spectra straightway. In non-relativistic limit, the eigenvalues lead the corresponding non-relativistic results. And the upper-left elements of the operators coincide with their non-relativistic counterpart accurately, while the lower-right ones take their free particle limits.

Since Ginocchio has found the $U(3)$ symmetry in 3D case with the harmonic oscillator potential, we can foretell our treatment can be generalized to the ND Dirac system to find the $SO(N+1)$ symmetry of the hydrogen atom and the $SU(N)$ symmetry of the harmonic oscillator. Recently, Alhaidaria *et. al.* [14] has revealed the equivalence between the Dirac equation and the Klein-Gordon equation with the equal scalar and vector potentials. This suggests the dynamical symmetries should exist in the spin-0 system.

Acknowledgments

We thank S. W. Hu for his suggestions in grammar. This work is supported in part by NSF of China (Grants No. 10575053 and No. 10605013) and Program for New Century Excellent Talents in University. The Project-sponsored by SRF for ROCS, SEM.

[1] W. Greiner and B. Müller, *Quantum Mechanics (Symmetries)* (Springer, 1994), p. 453.
[2] J. Bertrand, C. R. Acad. Sci. Paris. **77** (1873).
[3] W. Lenz, Z. Phys. **24**, 197 (1925).
[4] W. Pauli, Z. Phys. **36**, 336 (1926).
[5] D. M. Fradkin, American Journal of Physics **33**, 207 (1965).
[6] M. Moshinsky and A. Szczepaniak, J. Phys. A: Math. Gen. **22**, L817 (1989).
[7] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Butterworth-Heinemann, 1982).
[8] J. S. Bell and H. Ruegg, Nucl. Phys. B **98**, 151 (1975).
[9] A. L. Blokhin, C. Bahri, and J. P. Draayer, Phys. Rev. Lett. **74**, 4149 (1995).
[10] J. N. Ginocchio, Phys. Rev. C **69**, 034318 (2004).
[11] J. N. Ginocchio, Phys. Rev. Lett. **95**, 252501 (2005).
[12] J. N. Ginocchio, Phys. Rep. **414**, 165 (2005).
[13] X. L. Yang, S. H. Guo, F. T. Chan, K. W. Wong, and W. Y. Ching, Phys. Rev. A **43**, 1186 (1991).
[14] A. Alhaidaria, H. Bahloul, and A. Al-Hasan, Phys. Lett. A **349**, 87 (2006).